ELSEVIER

Journal of Pure and Applied Algebra 154 (2000) 3–13

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

The universal property of the multitude of trees

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Communicated by M. Tierney

Dedicated to Bill Lawvere

Abstract

A vital ingredient in the first author's definition of weak ω-category is his description, in terms of trees, of the free (strict) ω-category on a globular set. The induced monad on the category of globular sets shares many of the properties of the monoid monad (describable in terms of words) on the category of sets. Bénabou has shown how the simplicial category arises from the monoid monad. The present paper studies the object arising similarly from the ω-category monad. © 2000 Elsevier Science B.V. All rights reserved.

MSC: 18D05

Lawvere [13] pointed out that the category Δ, whose objects are finite ordinals and whose arrows are order-preserving functions, is the generic monoidal category containing a monoid. Let **Mon** be the category of monoids in the category **Set** of sets. Bénabou [6] remarked that the (simplicial) nerve of the category Δ is the standard resolution [2] of the terminal monoid via the comonad generated by the underlying functor **Mon** $\to$ **Set** and its left adjoint.

Let **Omcat** denote the category of ω-categories and let **Glob** denote the category of globular sets. In this note we announce a generic property of the category Ω whose nerve is the standard resolution of the terminal ω-category via the comonad generated by the underlying functor **Omcat** $\to$ **Glob** and its left adjoint. We give a concrete model for Ω in terms of trees. Furthermore, we make connections with the recent work of Joyal [11].

A *globular object* X in a category $\mathscr{X}$ is a sequence $(X_n)_{n\geq 0}$ of objects X_n together with arrows $s_n, t_n : X_{n+1} \to X_n$ such that $s_n \circ s_{n+1} = s_n \circ t_{n+1}$, $t_n \circ s_{n+1} = t_n \circ t_{n+1}$. Each

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PII: S0022-4049(99)00184-X

globular object X gives a diagram in $\mathscr{X}$.

$$\mathrel{\substack{\xrightarrow{s_3} \\ \xrightarrow[t_3]{}}} X_3 \mathrel{\substack{\xrightarrow{s_2} \\ \xrightarrow[t_2]{}}} X_2 \mathrel{\substack{\xrightarrow{s_1} \\ \xrightarrow[t_1]{}}} X_1 \mathrel{\substack{\xrightarrow{s_0} \\ \xrightarrow[t_0]{}}} X_0.$$

For $m < n$, we write $s_m, t_m : X_n \to X_m$ for the composite purely of arrows s_r, t_r, $m \leq r < n$, respectively. There is a category Glob $\mathscr{X}$ of globular objects where the arrows are morphisms of diagrams. For $m < n$, define the object $X_n \underset{m}{\times} X_n$ to be the following pullback (assuming $\mathscr{X}$ has the pullbacks).

$$\begin{tikzcd} X_n \underset{m}{\times} X_n \arrow[r, "p_2"] \arrow[d, "p_1"'] & X_n \arrow[d, "t_m"] \\ X_n \arrow[r, "s_m"'] & X_m \end{tikzcd}$$

Recall that an *ω-category* in $\mathscr{X}$ is a globular object X together with, for all $m < n$, arrows

$$\#_m : X_n \underset{m}{\times} X_n \to X_n, \qquad i_m : X_m \to X_n,$$

such that the diagram

$$X_n \underset{m}{\times} X_n \mathrel{\substack{\xrightarrow{p_2} \\ \xrightarrow{\#_m} \\ \xrightarrow{p_1}}} X_n \mathrel{\substack{\xrightarrow{s_m} \\ \xleftarrow{i_m} \\ \xrightarrow{t_m}}} X_m$$

is the truncation of the nerve of a category in $\mathscr{X}$ and, for all $m < k < n$, the arrows $\#_m$, i_m are functors in $\mathscr{X}$ for the category structures on $X_n \underset{m}{\times} X_n$, X_n, X_m determined by $\#_k$, i_k. Write Omcat $\mathscr{X}$ for the category of ω-categories in $\mathscr{X}$. Put

$$\mathbf{Glob} = \mathrm{Glob}\mathbf{Set} \quad \text{and} \quad \mathbf{Omcat} = \mathrm{Omcat}\mathbf{Set}.$$

Now, we describe the basic example of what Baez–Dolan [1] call "the microcosm principle". Traditionally, a *monoid* is a *set* with an associative unital binary operation. The "categorification" of monoid is monoidal category and we know that we can define monoid in any monoidal category. Mac Lane's coherence theorem [15] implies that every monoidal category is equivalent to a monoid in the cartesian monoidal category **Cat**.

A monoid M is a special ω-category, namely, one with $X_n = M$ for all $n \geq 1$, with $X_0 = 1$ and with s_m, t_m equal to the identity function of M for $m \geq 1$. This suggests generalisation of the microcosm principle to ω-categories.

The categorification of ω-category is *monoidal globular category* in the sense of the first author [5]. Furthermore, it is shown in [5] that every monoidal globular category is equivalent to an ω-category in **Cat**.

A monoidal globular category is a globular object $\mathscr{C}$ in **Cat** together with functors $i_m : \mathscr{C}_m \to \mathscr{C}_n$, $\#_m : \mathscr{C}_n \underset{m}{\times} \mathscr{C}_n \to \mathscr{C}_n$ such that $s_m \circ i_m$ and $t_m \circ i_m$ are identity functors and the equations

$$s_m(A \#_m B) = s_m A, \quad t_m(A \#_m B) = t_m A, \quad s_m(A \#_k B) = s_m A \#_k s_m B,$$
$$t_m(A \#_k B) = t_m A \#_k t_m B,$$

hold, and with globular natural isomorphisms

$$\alpha_m : (A \#_m B)\#_m C \xrightarrow{\sim} A\#_m(B\#_m C) \quad (\textit{associativity constraint}),$$

$$\lambda_m : i_m s_m A \#_m A \xrightarrow{\sim} A \quad (\textit{left unit constraint}),$$

$$\rho_m : A\#_m i_m s_m A \xrightarrow{\sim} A \quad (\textit{right unit constraint}),$$

$$\eta_{k,m} : (A\#_k B)\#_m(C\#_k D) \xrightarrow{\sim} (A\#_m C)\#_k(B\#_m D) \quad (\textit{interchange constraint})$$

for all $k < m < n$ and all $A, B, C, D \in \mathscr{C}_n$ for which the expressions make sense, such that certain diagrams commute. This leads us to a new concept in such a structure $\mathscr{C}$.

Definition. *A globular monoid A in* $\mathscr{C}$ consists of an object A_n of $\mathscr{C}_n$ for each $n \geq 0$ and arrows

$$m_r : A_n \#_r A_n \to A_n, \qquad i_r : A_r \to A_n$$

in $\mathscr{C}_n$ for all $0 \leq r < n$ such that

- $A = (A_n)_{n \geq 0}$ is a globular object of $\mathscr{C}$ (that is, $A_n = s_n(A_{n+1}) = t_n(A_{n+1})$ for all $n \geq 0$),
- for all $0 \leq r < n$, the triple (A_n, m_r, e_r) is a monoid in the monoidal category $\mathscr{C}_n$ with tensor product functor $\#_r$ and unit object $i_r(A_r)$, and,
- for all $0 \leq r < s < n$, the following two diagrams commute:

$$\begin{tikzcd}
(A_n \#_r A_n) \#_s (A_n \#_r A_n) \arrow[rr, "m_r \#_s m_r"] \arrow[d, "\eta"'] & & A_n \#_s A_n \arrow[dd, "m_s"] \\
(A_n \#_s A_n) \#_r (A_n \#_s A_n) \arrow[d, "m_s \#_r m_s"'] & & \\
A_n \#_r A_n \arrow[rr, "m_r"'] & & A_n
\end{tikzcd}$$

$$\begin{tikzcd}
i_s(A_n) \#_r i_s(A_n) \arrow[r, "\lambda = \rho"] \arrow[d, "e_s \#_r e_s"'] & i_s(A_n) \arrow[d, "e_s"] \\
A_n \#_r A_n \arrow[r, "m_r"'] & A_n
\end{tikzcd}$$

Alternatively, for readers familiar with operads in the sense of Batanin [3,4], a globular monoid is precisely an algebra for the terminal operad in $\mathscr{C}$.

Example 1. Each bicategory $\mathscr{B}$ gives a monoidal globular category $\mathscr{C}$ with $\mathscr{C}_0$ the discrete category of objects of $\mathscr{B}$, and with $\mathscr{C}_n$ $(n > 0)$ the category whose objects are the arrows of $\mathscr{B}$ and whose arrows are the 2-cells of $\mathscr{B}$. A globular monoid in $\mathscr{C}$ is precisely a monad in $\mathscr{B}$.

Example 2. Let $\mathscr{V}$ be a braided monoidal category. This gives a monoidal globular category $\mathscr{C}$ with $\mathscr{C}_0 = \mathscr{C}_1 = 1$ and $\mathscr{C}_n = \mathscr{V}$ for $n > 1$. A globular monoid in $\mathscr{C}$ is precisely a commutative monoid in $\mathscr{V}$.

Example 3. A globular monoid in the monoidal globular category **Span** (see [5] or [17]) of higher spans is precisely an ω-category (in **Set**).

The main object of study in this work is the *generic example* of globular monoid.

The forgetful functor **Omcat** $\to$ **Glob** is monadic. In particular, it has a left adjoint D_s whose description can be made explicitly in terms of trees (see [5] or [17]). The free ω-category $D_s\mathbf{1}$ on the terminal globular set $\mathbf{1}$ is precisely **Tree**. An element of the set $\mathbf{Tree}_n$ is a (plane) tree T of height n; that is, a functor $T : [n]^{\mathrm{op}} \to \Delta$, where $[n] = \{0, 1, 2, \dots, n\}$ as a linearly ordered set, such that $T(0) = [0]$. (Examples of trees can be found in many places [9].) So a tree of height n can be identified with an element of dimension n of the nerve of the category Δ. The trees $s_{n-1}(T)$ and $t_{n-1}(T)$ are both equal to the restriction of T to $[n-1] \subset [n]$. This defines **Tree** as a globular set.

The ω-category structure on **Tree** is explicitly described in [5,17], but we shall give another approach related to the work of Bénabou [6] from which it follows that the nerve of the monoidal category Δ is isomorphic to the simplicial monoid

$$\begin{tikzcd}[column sep=large] \cdots\ 1^{****} \arrow[r, shift left=4.5, "\mu_1{**}"] \arrow[r, shift left=1.5, "(\mu_{1*})^*"] \arrow[r, shift right=1.5, "(\mu_1)^{**}"] \arrow[r, shift right=4.5, "\tau^{***}"] & 1^{***} \arrow[r, shift left=3, "\mu_1{*}"] \arrow[r, "(\mu_1)^*"] \arrow[r, shift right=3, "\tau^{**}"] & 1^{**} \arrow[r, shift left=1.5, "\mu_1"] \arrow[r, shift right=1.5, "\tau^*"] & 1^*, \end{tikzcd}$$

where X^* is the free monoid on the set X, where $\mu_M : M^* \to M$ takes each word in elements of the monoid M to the product of the letters in the word, and where $\tau : X \to 1$ is the unique function from the set X into the one element set 1. So $\mathbf{Tree}_n$ is isomorphic to the n-fold iterate of the functor $(\ )^* : \mathbf{Set} \to \mathbf{Set}$ applied to the terminal set 1; for example, $\mathbf{Tree}_1 \cong 1^*$ is isomorphic to the set $\mathbb{N}$ of natural numbers. Since $\mathbf{Tree}_{n-k}$ is isomorphic to a free monoid, it obtains a monoid structure by transport; this monoid is a one object category and so has a nerve as shown below.

$$\begin{tikzcd}[column sep=large] \mathbf{Tree}_{n-k} \times \mathbf{Tree}_{n-k} \arrow[r, shift left=3, "p_2"] \arrow[r, "m"] \arrow[r, shift right=3, "p_1"] & \mathbf{Tree}_{n-k} \arrow[r, shift left=1.5, "\tau"] \arrow[r, shift right=1.5, "\tau"] & 1 \end{tikzcd}$$

Since $(\ )^* : \mathbf{Set} \to \mathbf{Set}$ preserves pullbacks, we can apply it k times to the above nerve and obtain a category structure whose underlying graph is shown below (recall that $s_k = t_k$).

$$\mathbf{Tree}_n \underset{t_k}{\overset{s_k}{\rightrightarrows}} \mathbf{Tree}_k$$

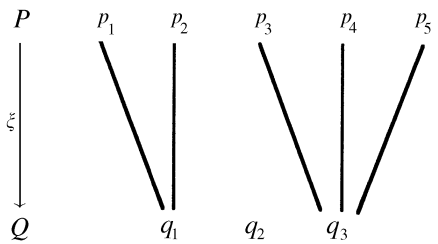
The compositions $\#_k$, $0 \le k < n$, coming from these category structures, determine the ω-category structure on **Tree**.

The free ω-category $D_s X$ on any globular set X consists of "globular pasting diagrams in X". These globular pasting diagrams are parametrized by trees. We need to see how a tree T gives rise to a globular set $|T|$; then a T-parametrized globular pasting diagram in X will be a globular function $|T| \to X$. A tree T of height n is a diagram

$$1 = T(0) \xleftarrow{\xi_1} T(1) \xleftarrow{\xi_2} T(2) \xleftarrow{\xi_3} \cdots \xleftarrow{\xi_n} T(n)$$

of finite ordinals $T(i)$ and order-preserving functions ξ_i.

In order to describe $|T|$ (which was called T^* in [5]), we shall make use of a general construction of a graph $G(\xi)$ from a single order-preserving function $\xi : P \to Q$ between linearly ordered sets P, Q. First, consider the linear order on the disjoint union $P + Q$ which takes each $q \in Q$ to be a maximal element added to the linearly ordered fibre $\xi^{-1}(q) \subseteq P$. (For example, if ξ is the function



then $P + Q$ has the order $p_1 < p_2 < q_1 < q_2 < p_3 < p_4 < p_5 < q_3$.)

Take the set $G(\xi)_0$ of vertices to be the disjoint union $P + Q$ and take the set $G(\xi)_1$ of edges to be P. The source and target functions $s, t : G(\xi)_1 \to G(\xi)_0$ are given by $s(p) = p$ and $t(p) =$ the successor of p in the order on $P + Q$. (In our example, $G(\xi)$ is the directed graph $p_1 \to p_2 \to q_1 \ q_2 \ p_3 \to p_4 \to p_5 \to q_3$.)

Now we can give an inductive definition of $|T|$. For a tree T of height 0, we take $|T|$ to be 1 in dimension 0 and empty in higher dimensions. For a tree T of height 1, we take $|T|$ to be the 1-skeletal globular set $G(\xi_1 : T(1) \to 1)$. For a tree T of height n, we shall recursively define an n-skeletal globular set $|T|$ whose elements of dimension n are the vertices of T of height n (that is, $|T|_n = T(n)$). So we can suppose we already have the $(n-1)$-skeletal globular set $|s_{n-1}(T)|$ $(=|t_{n-1}(T)|)$ with $|s_{n-1}(T)|_{n-1} = s_{n-1}(T)(n-1) = T(n-1)$. The $(n-2)$-skeleton of $|T|$ agrees with the $(n-2)$-skeleton of $|s_{n-1}(T)|$. The graph

$$|T|_n \underset{t_{n-1}}{\overset{s_{n-1}}{\rightrightarrows}} |T|_{n-1}$$

is just $G(\xi_n : T(n) \to T(n-1))$; so that $|T|_n = T(n)$ and $|T|_{n-1} = T(n) + T(n-1)$. All that remains is to provide the functions

$$|T|_{n-1} \underset{t_{n-2}}{\overset{s_{n-2}}{\rightrightarrows}} |T|_{n-2} = |s_{n-1}T|_{n-2};$$

on $T(n-1)$ these agree with the source, target functions of $s_{n-1}(T)$ while on $T(n)$ they amount to first applying $\xi_n : T(n) \to T(n-1)$ and then following with the source, target functions of $s_{n-1}(T)$.

An easy example of this construction is for the tree U_n: this is the tree of height n with $U_n(r) = 1$ for all $r \leq n$. Then the globular set $|U_n|$ is n-skeletal, has two elements in each dimension $< n$, and one element in dimension n. The first elements in each dimension $< n$ provide the sources of the elements of higher dimension, and the second elements the targets. Clearly $|s_{n-1}U_n| = U_{n-1}$.

For each tree T of height n, we have two globular functions $\sigma_{n-1}, \tau_{n-1} : |s_{n-1}T| \to |T|$ which are both identity functions in dimensions $< n-1$, are the unique function $\emptyset \to T(n)$ in dimension n, and in dimension $n-1$ the function $(\tau_{n-1})_{n-1} : T(n-1) \to T(n) + T(n-1)$ is the second coprojection while $(\sigma_{n-1})_{n-1}$ takes $q \in T(n-1)$ to the first element of $T(n)$ in the fibre of ξ_n over q or to q itself if the fibre is empty. Indeed, we obtain an n-truncated coglobular globular set $||T||$ as shown below.

$$\cdots \rightrightarrows |s_{n-3}T| \underset{\tau_{n-3}}{\overset{\sigma_{n-3}}{\rightrightarrows}} |s_{n-2}T| \underset{\tau_{n-2}}{\overset{\sigma_{n-2}}{\rightrightarrows}} |s_{n-1}T| \underset{\tau_{n-1}}{\overset{\sigma_{n-1}}{\rightrightarrows}} |T|$$

In particular, the globular functions $\sigma_{n-1}, \tau_{n-1} : |U_{n-1}| \to |U_n|$ are the identity functions in dimensions $< n-1$ while they take the element of $|U_{n-1}|$ of dimension $n-1$ to the first and last elements of $|U_n|$, respectively.

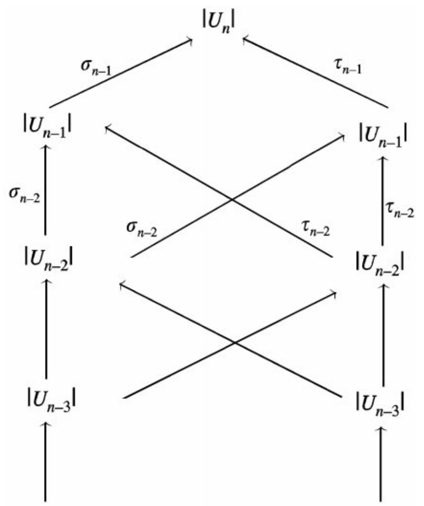
It can be shown that, if T, T' are k-composable trees to height n, the following square is a pushout of globular sets:

$$\begin{tikzcd}
|s_k T| \arrow[r, "\tau_k"] \arrow[d, "\sigma_k"'] & |T'| \arrow[d] \\
|T| \arrow[r] & |T \#_k T'|
\end{tikzcd}$$

A more conceptual description of $||T||$ is as follows. The category **Glob** of globular sets has pushouts formed pointwise. So we can form the monoidal globular category

$$\mathrm{Cospan}(\mathbf{Glob}) = \mathrm{Span}(\mathbf{Glob}^{\mathrm{op}})^{\mathrm{op}}.$$

There is a distinguished globular element of Cospan(**Glob**) which is the following cospan in dimension n:



This determines a global element $\mathbf{1} \to$ Cospan(**Glob**), and hence, by the freeness of **Tree**, determines a monoidal globular functor

$$|| - || : \mathbf{Tree} \to \mathrm{Cospan}(\mathbf{Glob}),$$

the value of this monoidal globular functor at the tree T is $||T||$ as defined before.

For each globular set X, we can now describe the free ω-category D_sX on X. The elements of the set $(D_sX)_n$ are pairs (T,h) where T is a tree of height n and $h : |T| \to X$ is a globular function. The globular structure on D_sX is given by

$$s_k(T,h) = (s_k(T), |s_k(T)| \xrightarrow{\sigma_k} |T| \xrightarrow{h} X), \quad t_k(T,h) = (s_k(T), |s_k(T)| \xrightarrow{\tau_k} |T| \xrightarrow{h} X).$$

The ω-category structure on D_sX comes from the pushout property of $|T \#_k T'|$: the equality $s_k(T,h) = t_k(T',h')$ implies the existence of a globular function $h \#_k h'$: $|T \#_k T'| \to X$ whose restriction to $|T|$, $|T'|$ is h, h' respectively; so we put

$$(T,h) \#_k (T',h') = (T \#_k T', h \#_k h').$$

We ambiguously denote by $(-)^*$ the monad on **Glob** generated by the forgetful functor **Omcat** $\to$ **Glob** and its left adjoint D_s : **Glob** $\to$ **Omcat**. It follows from the above explicit construction that this monad $(-)^*$: **Glob** $\to$ **Glob** has a lot in common with the monad $(-)^*$: **Set** $\to$ **Set**. Such properties have been studied in various degrees of abstraction by Bénabou [6], Cockett [7,8], Hermida [10], Kelly [12] and

Leinster [14]. The endofunctor of the monad preserves pullbacks and the unit and multiplication are shape (or "cartesian") transformations.

In particular, it follows that the simplicial ω-category

$$\cdots\ 1^{****} \begin{array}{c} \xrightarrow{\mu_1**} \\ \xrightarrow{(\mu_{1*})^*} \\ \xrightarrow{(\mu_1)^{**}} \\ \xrightarrow{\tau^{***}} \end{array} 1^{***} \begin{array}{c} \xrightarrow{\mu_{1*}} \\ \xrightarrow{(\mu_1)^*} \\ \xrightarrow{\tau^{**}} \end{array} 1^{**} \begin{array}{c} \xrightarrow{\mu_1} \\ \xrightarrow{\tau^*} \end{array} 1^*,$$

is the nerve of a category Ω in **Omcat**. Equivalently, Ω is an ω-category in **Cat**; that is, a strict monoidal globular category. The objects of Ω_n are elements of $(1^*)_n = \mathbf{Tree}_n$; that is, trees of height n. An arrow $w : S \to T$ in Ω_n is a globular function $w : |T| \to$ **Tree** such that

$$\mu_1(T, w) = S.$$

This last equation means that the pasted composite of the globular diagram (T, w) in **Tree** is equal to S. The globular structure $s_{n-1}, t_{n-1} : \Omega_n \to \Omega_{n-1}$ is given by taking $w : S \to T$ to $s_{n-1}(w), t_{n-1}(w) : s_{n-1}(S) {\to} s_{n-1}(T)$ which are the composites of $w : |T| \to$ **Tree** with $\sigma_{n-1}, \tau_{n-1} : |s_{n-1}(T)| \to |T|$, respectively. The ω-category composition $\#_k : \Omega_n \underset{k}{\times} \Omega_n \to \Omega_n$ takes a pair $(w : S \to T, w' : S' \to T')$ with $s_k(S) = s_k(S'), s_k(T) = s_k(T'), w \circ \tau_k = w' \circ \sigma_k$ to $w \#_k w' : S \#_k S' \to T \#_k T'$ (which makes sense because the pasting operation μ_1 is an ω-functor). Finally, $i_k : \Omega_k \to \Omega_n$ takes a tree of height k to the same tree regarded as of height $n > k$.

The sequence $U = (U_n)$ of trees U_n provides a globular monoid in the monoidal globular category Ω. To describe the arrow $m_k : U_n \#_k U_n \to U_n$ in Ω_n we must provide a globular function $m_k : |U_n| \to$ **Tree** which pastes to give $U_n \#_k U_n$; but such a globular function is determined by the requirement that it take the single element of $|U_n|$ of dimension n to the tree $M_n^k = U_n \#_k U_n$ of height n. Similarly, to describe the arrow $e_k : i_k(U_k) \to U_n$ in Ω_n we must provide a globular function $e_k : |U_n| \to$ **Tree** which pastes to give $i_k(U_k)$; but such a globular function is determined by the requirement that it take the single element of $|U_n|$ of dimension n to the tree $i_k(U_k)$ of height n. Now we can state our main result which admits a proof along the lines of that for $(\Delta, 1)$.

Theorem 1. *(Ω, U) is the generic monoidal globular category containing a globular monoid. That is, given any globular monoid X in a monoidal globular category $\mathscr{X}$, there exists a monoidal globular functor $F : \Omega \to \mathscr{X}$ which is unique up to isomorphism with the property that $F(U) \cong X$.*

The connection of this work with that of Joyal [11] was understood during discussions with André Joyal in Montréal (September–October 1997). A *morphism* $f : S {\to} T$

of (*plane*) *trees* S, T is a commutative diagram

$$
\begin{tikzcd}
1 = S(0) \arrow[d, "f_0"'] & S(1) \arrow[l, "\xi_1"'] \arrow[d, "f_1"'] & S(2) \arrow[l, "\xi_2"'] \arrow[d, "f_2"'] & \cdots \arrow[l, "\xi_3"'] & S(n) \arrow[l, "\xi_n"'] \arrow[d, "f_n"'] \\
1 = T(0) & T(1) \arrow[l, "\xi_1"'] & T(2) \arrow[l, "\xi_2"'] & \cdots \arrow[l, "\xi_3"'] & T(n) \arrow[l, "\xi_n"']
\end{tikzcd}
$$

in **Set** such that each of the functions f_k preserves the linear order in the fibres of the functions ξ_k. (Obviously, this is weaker than asking that the diagram should commute in Δ.)

Theorem 2. *The category Ω_n is isomorphic to the category of* (*plane*) *trees of height n and their morphisms.*

Proof. Let us write $\mathbf{JTree}_n$ for the category of plane trees of height n and of their morphisms. Joyal [11] has defined an obvious strict monoidal globular category structure **JTree** on the family $\mathbf{JTree}_n$, $n \geq 0$. It is easy to see that a globular monoid U in **JTree** is given by the U_n. By the universal property given in Theorem 1, there exists a monoidal globular functor

$$\Phi : \Omega \to \mathbf{JTree},$$

which is the identity on objects.

Using an idea from [5], we can define a monoidal globular functor

$$\Psi : \mathbf{JTree} \to \Omega,$$

in the other direction, which is also the identity on objects. To define it on morphisms, let $f : S \to T$ be a morphism in $\mathbf{JTree}_n$ and let v be a vertex of T of height r. There exists a unique morphism $u_v : i_r(U_r) \to T$ in $\mathbf{JTree}_n$ which maps the vertex of height r of $i_r(U_r)$ to v. Pulling back F along u_v gives us a new tree S_v. It is not difficult to check that S_v, $v \in T$, is actually a diagram $\Psi(f)$ of trees whose pasted composite is S, and that Ψ is a monoidal globular functor taking U to U.

All that remains to be proved is that the composite $\Phi\Psi$ is the identity. But this follows readily from an inductive argument using the canonical decomposition of the domain tree T and the following simple disjointness observation:

For every morphism $f : S \to T_1 \#_0 T_2$ in **JTree**, *there exist unique morphisms*

$$f_1 : S \to T_1, \quad f_2 : S \to T_2$$

such that $f = f_1 \#_0 f_2$. $\square$

The category we have called Δ is sometimes called the “algebraic simplex category”. Our category Ω is a higher dimensional; version of Δ. There is also the “topological simplex category” Δ_t which is the full subcategory of Δ consisting of the non-empty linearly ordered sets $[n]$, $n \geq 0$. The starting point of Joyal [11] is the duality between

Δ_t and the category of finite intervals (also see [16]); a higher dimensional version of the category of finite intervals is constructed and consists of finite "disks". This leads to Joyal's category Θ which is isomorphic to the dual of the category of disks and is a higher dimensional version of Δ_t.

This begs the question of how Ω and Θ are related? Let us look again at obtaining Δ_t from Δ. We have already said that we can regard the objects of Δ as elements of 1^*, or as trees of height 1. There is a functor $\Delta(-,[1]) : \Delta^{\mathrm{op}} \to \mathbf{Cat}$ taking the tree T of height 1 with n leaves to the linearly ordered set $[n]$ regarded as a category, for all $n \geq 0$. This category $[n]$ can also be regarded as the free category on the graph $|T|$. At any rate, Δ_t is precisely the full image of the functor $\Delta(-,[1]) : \Delta^{\mathrm{op}} \to \mathbf{Cat}$; that is, the objects of Δ_t are the natural numbers $n \geq 0$ and the arrows $n \to n'$ are the functors $[n] \to [n']$.

Similarly, there is a functor $\Omega^{\mathrm{op}} \to \mathbf{Omcat}$ whose value at a tree T is the free ω-category $D_s|T|$ on the globular set $|T|$. The full image of this functor is Θ: the objects are the trees T and the arrows $T \to T'$ are the ω-functors $D_s|T| \to D_s|T'|$.

Joyal's program includes defining weak ω-categories in terms of their "nerves" which he conceives as functors $\Theta^{\mathrm{op}} \to \mathbf{Set}$. At least we can now see how to obtain nerves, in this sense, of strict ω-categories since we have a fully faithful functor $\mathscr{J} : \Theta \to \mathbf{Omcat}$ given by $\mathscr{J}(T) = D_s|T|$, and hence, a corresponding "singular functor" $N : \mathbf{Omcat} \to [\Theta^{\mathrm{op}}, \mathbf{Set}]$ into the presheaf category on Θ; that is, for each ω-category A, the Joyal-nerve of A is given by

$$N(A) = \mathbf{Omcat}(\mathscr{J}(-), A).$$

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